

# Noncommutative instantons on $d = 2n$ planes from matrix models

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## Abstract

In the case of an invertible coordinate commutator matrix  $\theta_{ij}$ , we derive a general instanton solution of the noncommutative gauge theories on  $d = 2n$  planes given in terms of  $n$  oscillators.

# 1 Introduction

The concept of noncommutative geometry [1]-[2] has nowadays acquired a central role in the study of possible extensions of gauge theories. Plausible reasonings based on quantum mechanics and general relativity imply that it is necessary, at least at Planck scale, to replace space-time coordinates by some noncommutative structure [3]-[4].

Many non perturbative aspects of noncommutative gauge theories have recently been explored; between them we can recall the Morita equivalence of the noncommutative torus, and the non perturbative structure of noncommutative gauge theories, i.e. the instantons ( in Euclidean space-time ) [5]-[6]-[7]-[8]-[9]-[10].

The first examples of noncommutative instantons were given by Nekrasov and Schwarz [7] who modified the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [11] to the noncommutative case, and showed that on noncommutative  $R^4$  non singular instantons exist even for the  $U(1)$  gauge group.

It is interesting to observe that introducing noncommutativity - which from the point of view of the star product corresponds to adding a complicated set of higher derivative interactions - can in fact greatly simplify the construction of soliton solutions.

Our point of view is searching a new systematic method of classifying soliton solutions starting from the Matrix model approach, instead of the twistor approach of Nekrasov and Schwarz [12]-[13]-[14]-[15].

It is well known that matrix model, suitably defined, encodes the noncommutative gauge theory on a  $d = 2n$  plane, by expanding the matrices  $X_i$  as a sum of a background ( representing the noncommutative coordinate system ) and fluctuations, which represent the gauge connection. In this paper we explore, by using different matrix models, i.e. the unitary matrix model and the hermitian one, the characterization of soliton solutions in terms of constraints in the matrices  $X_i$ , and solve them with an easy method.

The commonly accepted definition of soliton [16]-[17]-[18]-[19]-[20]-[21]-[22]-[23]-[24]-[25]-[26]-[27] is by the configuration

$$U^\dagger X^i U \tag{1.1}$$

obtained with a quasi-unitary operator  $U$  that satisfies

$$UU^\dagger = 1 \quad U^\dagger U = 1 - P_0 \tag{1.2}$$

where  $P_0$  is a finite projection operator.

In this article, instead of searching the general solution of this operator  $U$  directly in  $2n$ -dimensions, we apply the duality transformation [28]-[29] that maps a  $2n$ -dimensional plane in two dimensions, since then the quasi-unitary operator  $U$  is known to be simply the shift operator. Using back again the duality transformation one can easily write down general formulas for the quasi-unitary operator  $U$  and instanton configuration  $X_i$  in every even dimension.

The solution is based on a representation in terms of  $n$ -oscillators in  $d = 2n$  dimensions, whose Hilbert space is the tensorial product of  $n$  Hilbert spaces of one oscillator.

The duality transformation is based on the isomorphism between  $n$  quantum numbers and only one quantum number, which allows us to reduce the general solution in  $d = 2n$  dim. to a solution with a single oscillator, i.e. in  $d = 2$  dimensions.

## 2 Matrix models and noncommutative gauge theories

In this paper we study two types of matrix models, i.e. the unitary matrix model and the hermitian matrix model, and their relation with noncommutative gauge theory in  $d = 2n$  dimensions.

Let us start with  $d = 2$  and the unitary matrix model

$$S = \frac{\beta_0}{\theta_0^2} \text{Tr}[(U_1 U_2 - e^{-i\theta_0} U_2 U_1)(h.c.)] \quad \theta_0 = \frac{2\pi}{N} \quad (2.1)$$

where  $U_1, U_2$  are two independent unitary matrices  $N \times N$ . In order to define a non-commutative  $U(1)$  gauge theory in two dimensions on a noncommutative plane, a certain  $N \rightarrow \infty$  limit must be taken.

The model has a positive definite action  $S$ , with its minimum reached at

$$U_1^{(0)} U_2^{(0)} = e^{-i\theta_0} U_2^{(0)} U_1^{(0)} \quad (2.2)$$

which are the commutation relations of a noncommutative torus. In order to define the fluctuation around this background solution we define:

$$U_1 = U_1^{(0)} e^{i\sqrt{\theta\theta_0} a_1(U_1^{(0)}, U_2^{(0)})} \quad U_2 = U_2^{(0)} e^{i\sqrt{\theta\theta_0} a_2(U_1^{(0)}, U_2^{(0)})} \quad (2.3)$$

where  $a_1$  and  $a_2$  are two hermitian matrices, that can be developed, as in a Fourier series, in terms of the two basic matrices  $U_1^{(0)}, U_2^{(0)}$ . By using simply the commutation relations, one finds:

$$\begin{aligned} U_1 U_2 &= U_1^{(0)} U_2^{(0)} e^{i\sqrt{\theta\theta_0}a_1(U_1^{(0)}e^{-i\theta_0}, U_2^{(0)})} e^{i\sqrt{\theta\theta_0}a_2(U_1^{(0)}, U_2^{(0)})} \\ U_2 U_1 &= U_2^{(0)} U_1^{(0)} e^{i\sqrt{\theta\theta_0}a_1(U_1^{(0)}, U_2^{(0)}e^{i\theta_0})} e^{i\sqrt{\theta\theta_0}a_2(U_1^{(0)}, U_2^{(0)})}. \end{aligned} \quad (2.4)$$

Now we consider the  $N \rightarrow \infty$  limit ( i.e.  $\theta_0 \rightarrow 0$  ). In this case it is possible to define two noncommutative coordinates from the commutation relations (2.2)

$$U_1^{(0)} = e^{i\sqrt{\frac{\theta_0}{\theta}}\hat{x}_2} \quad U_2^{(0)} = e^{-i\sqrt{\frac{\theta_0}{\theta}}\hat{x}_1} \quad \Rightarrow \quad [\hat{x}_1, \hat{x}_2] = i\theta. \quad (2.5)$$

In terms of them, the fluctuations can be recast as follows

$$\begin{aligned} U_1 U_2 &= U_1^{(0)} U_2^{(0)} e^{i\sqrt{\theta\theta_0}a_1(\hat{x}_1, \hat{x}_2 - \sqrt{\theta\theta_0})} e^{i\sqrt{\theta\theta_0}a_2(\hat{x}_1, \hat{x}_2)} \\ U_2 U_1 &= U_2^{(0)} U_1^{(0)} e^{i\sqrt{\theta\theta_0}a_1(\hat{x}_1 - \sqrt{\theta\theta_0}, \hat{x}_2)} e^{i\sqrt{\theta\theta_0}a_2(\hat{x}_1, \hat{x}_2)}. \end{aligned} \quad (2.6)$$

By expanding the exponentials and keeping only the relevant terms, one finds

$$\begin{aligned} U_1 U_2 - e^{-i\theta_0} U_2 U_1 &= U_1^{(0)} U_2^{(0)} [i\sqrt{\theta\theta_0}(a_1(\hat{x}_1, \hat{x}_2 - \sqrt{\theta\theta_0}) - a_1(\hat{x}_1, \hat{x}_2)) \\ &\quad - \frac{\theta\theta_0}{2}(a_1^2 + a_2^2 + 2a_1 a_2) - i\sqrt{\theta\theta_0}(a_2(\hat{x}_1 - \sqrt{\theta\theta_0}, \hat{x}_2) - a_2(\hat{x}_1, \hat{x}_2)) \\ &\quad + \frac{\theta\theta_0}{2}(a_1^2 + a_2^2 + 2a_2 a_1)] + O(\sqrt{\theta_0^3}) = \\ &= U_1^{(0)} U_2^{(0)} (-i\theta\theta_0(\partial_2 a_1 - \partial_1 a_2) - \theta\theta_0[a_1, a_2]) \\ &= i\theta\theta_0 U_1^{(0)} U_2^{(0)} (\partial_1 a_2 - \partial_2 a_1 + i[a_1, a_2]) = i\theta\theta_0 U_1^{(0)} U_2^{(0)} F_{12} + O(\sqrt{\theta_0^3}). \end{aligned} \quad (2.7)$$

Strictly speaking, defining a derivative with respect to a noncommutative coordinate is not properly defined. One should pass from operators to their symbols, i.e. commutative functions, as in [28], and introduce, instead of the products of operators the star product of their corresponding symbols; however we will never need to use the symbols and we will omit them in the following.

The action which survives in the  $N \rightarrow \infty$  limit is the  $U(1)$  noncommutative Yang-Mills action:

$$S = \lim_{\theta_0 \rightarrow 0} \frac{\beta_0}{\theta_0^2} \text{Tr}(U_1 U_2 - e^{-i\theta_0} U_2 U_1)(h.c.) = \beta_0 \theta^2 \text{Tr} F_{12}^2. \quad (2.8)$$

We have reached the noncommutative plane from a careful parametrization of the commutation relation (2.2), which represents instead a noncommutative torus as a background.

Note the importance of the presence of the scaling factor  $N^2$  in front of the action  $S$ , to reach a finite action in the  $N \rightarrow \infty$  limit.

It is not difficult to relate this model to a hermitian matrix model directly. The two independent unitary matrices  $U_1, U_2$  can be parameterized in terms of two infinite-dimensional hermitian matrices as follows

$$U_1 = e^{i\sqrt{\theta_0\theta}X_1} \quad U_2 = e^{i\sqrt{\theta_0\theta}X_2}. \quad (2.9)$$

Then the commutation relation

$$U_1 U_2 - e^{-i\theta_0} U_2 U_1 = -\theta\theta_0([X_1, X_2] - \frac{i}{\theta}) + O(\sqrt{\theta_0^3}) \quad (2.10)$$

is equivalent to a commutation relation between the two infinite-dimensional hermitian matrices.

Therefore, in the limit  $\theta_0 \rightarrow 0$ , the action  $S$  becomes:

$$S = -\beta_0 \theta^2 \text{Tr}([X_1, X_2] - \frac{i}{\theta})^2 \quad (2.11)$$

which again is positive definite and reaches its minimum on the background

$$[X_1, X_2] = \frac{i}{\theta} \quad (2.12)$$

which is satisfied by a noncommutative plane, with the position

$$X_1 = \frac{\hat{x}_2}{\theta} \quad X_2 = -\frac{\hat{x}_1}{\theta} \quad \Rightarrow [\hat{x}_1, \hat{x}_2] = i\theta. \quad (2.13)$$

The corresponding equations of motion are identical to those of the standard hermitian matrix model given by :

$$S_0 = -\beta_0 \theta^2 \text{Tr}[X_1, X_2]^2 \quad (2.14)$$

i.e.

$$[X_j, [X_i, X_j]] = 0 \quad i, j = 1, 2. \quad (2.15)$$

To find the direct connection between hermitian matrix model and Yang-Mills action, it is enough to develop the hermitian matrices as a background + fluctuations as follows :

$$X_1 = \frac{\hat{x}_2}{\theta} + a_1(\hat{x}_1, \hat{x}_2) \quad X_2 = -\frac{\hat{x}_1}{\theta} + a_2(\hat{x}_1, \hat{x}_2) \quad (2.16)$$

from which

$$[X_1, X_2] - \frac{i}{\theta} = -iF_{12} \quad (2.17)$$

and the action becomes

$$S = -\beta_0 \theta^2 \text{Tr} \left( [X_1, X_2] - \frac{i}{\theta} \right)^2 = \beta_0 \theta^2 \text{Tr} F_{12}^2. \quad (2.18)$$

### 3 Instantons in two dimensions

Let us recall the action for two-dimensional unitary matrix model

$$S = \frac{\beta_0}{\theta_0^2} \text{Tr} [(U_1 U_2 - e^{-i\theta_0} U_2 U_1)(h.c.)] \quad \theta_0 = \frac{2\pi}{N}. \quad (3.1)$$

The search for instantons of the noncommutative planes is driven by two criteria. The first is that an instanton is a solution of the equations of motion of  $S$ , and the second is that this solution must give a finite contribution to the action, also in the  $N \rightarrow \infty$  limit.

The classical equations of motion, obtained from a variation of the basic elements of the theory,  $U_1$  and  $U_2$ , are:

$$[U_i, V - V^\dagger] = 0 \quad (3.2)$$

where  $V = e^{-i\theta_0} U_1 U_2 U_1^\dagger U_2^\dagger$ .

Between all the possible solutions to equation (3.2), we find that the instantons are characterized by the matrices of the form:

$$\tilde{U}_1 = \begin{pmatrix} U_1^{(0)} & 0 \\ 0 & U_1^{(1)} \end{pmatrix} \quad \tilde{U}_2 = \begin{pmatrix} U_2^{(0)} & 0 \\ 0 & U_2^{(1)} \end{pmatrix} \quad (3.3)$$

where  $U_1^{(0)}$  and  $U_2^{(0)}$  are two  $(N - k) \times (N - k)$  unitary matrices satisfying

$$U_1^{(0)} U_2^{(0)} = e^{\frac{2i\pi}{N-k}} U_2^{(0)} U_1^{(0)} \quad (3.4)$$

and  $U_1^{(1)}, U_2^{(1)}$  are two  $k \times k$  diagonal unitary matrices.

Computing the action (3.1) on the solution (3.3) is not a difficult task:

$$S = \frac{\beta_0}{\theta_0^2} [2(1 - \cos(\frac{2\pi}{N} - \frac{2\pi}{N-k}))(N - k) + 2(1 - \cos \frac{2\pi}{N})k]. \quad (3.5)$$

The first term is order  $O(\frac{1}{N})$  and vanishes in the  $N \rightarrow \infty$  limit, while the second term is finite and gives as contribution:

$$S = \beta_0 k. \quad (3.6)$$

Such class of solution has a counterpart in the hermitian matrix model. Recall the action of the hermitian matrix model

$$S = -\frac{1}{2g^2} \text{Tr}([X_1, X_2] - \frac{i}{\theta})^2. \quad (3.7)$$

It is not difficult to extract from (3.3) that in the  $N \rightarrow \infty$  limit :

$$\lim_{\theta_0 \rightarrow 0} \frac{1}{\theta_0} (\tilde{U}_1 \tilde{U}_2 - e^{-i\theta_0} \tilde{U}_2 \tilde{U}_1) = \lim_{\theta_0 \rightarrow 0} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1-e^{-i\theta_0}}{\theta_0} \end{pmatrix} U_1^{(1)} U_1^{(2)} = i U_1^{(1)} U_1^{(2)} \begin{pmatrix} 0 & 0 \\ 0 & P_0 \end{pmatrix} \quad (3.8)$$

this commutation relation is equivalent to a  $k \times k$  projector  $P_0$ , apart from an irrelevant phase.

Since

$$\frac{1}{\theta_0} (\tilde{U}_1 \tilde{U}_2 - e^{-i\theta_0} \tilde{U}_2 \tilde{U}_1) = -\theta([X_1, X_2] - \frac{i}{\theta}) + O(\sqrt{\theta_0^3}) \quad (3.9)$$

we derive, that on an instanton solution, the following relation holds:

$$[X_1, X_2] = \frac{i}{\theta}(1 - P_0). \quad (3.10)$$

This condition will be the starting point of this article to characterize an instanton solution on a noncommutative plane. The instanton number is related to the rank  $k$  of the projector  $P_0$ , as it is clear from the evaluation of the solution (3.3) into the action :

$$S = -\frac{2}{g^2}\left(\frac{i}{\theta}\right)^2 \text{Tr} P_0 = \frac{1}{2g^2\theta^2}k. \quad (3.11)$$

Notice that, with a constant shift of the action (3.7) with respect to the usual hermitian matrix model, the evaluation of the action is null on the vacuum solution and finite on the instanton solution.

Notice also that both the action and the Chern class of  $F_{12}$  are linear in the instanton number, and that the Yang-Mills curvature  $F_{12}$  is a projector:

$$F_{12} = P_0. \quad (3.12)$$

The two dimensional plane is made by two coordinates which are promoted to operators acting on a Hilbert space. The Hilbert space is equivalent to the usual one oscillator Hilbert space, where the coordinates  $\hat{x}_1, \hat{x}_2$  are given in terms of oscillator rising and lowering operators:

$$\begin{aligned} a &= \sqrt{\frac{1}{2\theta}}(\hat{x}_1 + i\hat{x}_2) & \bar{a} &= \sqrt{\frac{1}{2\theta}}(\hat{x}_1 - i\hat{x}_2) \\ a|n\rangle &= \sqrt{n}|n-1\rangle & \bar{a}|n\rangle &= \sqrt{n+1}|n+1\rangle. \end{aligned} \quad (3.13)$$

Therefore the vacuum is characterized by the solution

$$\begin{aligned} X^1 + iX^2 &= -\frac{i}{\theta}(\hat{x}_1 + i\hat{x}_2) = -i\sqrt{\frac{2}{\theta}}a = -i\sqrt{\frac{2}{\theta}}\sum_{n=0}^{\infty}\sqrt{n+1}|n\rangle\langle n+1| \\ X^1 - iX^2 &= i\sqrt{\frac{2}{\theta}}\bar{a} = i\sqrt{\frac{2}{\theta}}\sum_{n=0}^{\infty}\sqrt{n+1}|n+1\rangle\langle n|. \end{aligned} \quad (3.14)$$

Instead, in the Hilbert space, the operator that defines the configuration of the  $k$ -instanton is given by :



$$\begin{aligned}
X^1 + iX^2 &= -i\sqrt{\frac{2}{\theta}}A_k & X^1 - iX^2 &= i\sqrt{\frac{2}{\theta}}\bar{A}_k \\
A_k &= \sum_{n_1=0}^{k-1} c_{n_1}|n_1\rangle\langle n_1| + \sum_{n_1=k}^{\infty} \sqrt{n_1+1-k}|n_1\rangle\langle n_1+1| \\
\bar{A}_k &= \sum_{n_1=0}^{k-1} \bar{c}_{n_1}|n_1\rangle\langle n_1| + \sum_{n_1=k}^{\infty} \sqrt{n_1+1-k}|n_1+1\rangle\langle n_1|
\end{aligned} \tag{3.15}$$

where

$$[A_k, \bar{A}_k] = 1 - P_0. \tag{3.16}$$

Till now we have discussed the case of the gauge group  $U(1)$ . It is possible to describe with the same method the instanton solution of  $U(2)$  gauge group, or in general the  $U(n)$  gauge group, by enlarging the Hilbert space one-oscillator to the base

$$|n; a\rangle \quad a = 0, 1 \quad \text{for } U(2) \quad \text{or} \quad a = 0, 1, \dots, n-1 \quad \text{for } U(n). \tag{3.17}$$

Then a  $k$ -instanton solution of  $U(2)$  can be characterized by the complex lowering oscillator operator:

$$\tilde{A}_k = \sum_{a=0}^1 \left( \sum_{n_1=0}^{k-1} c_{n,a} |n_1; a\rangle\langle n_1 : a| + \sum_{n_1=k}^{\infty} \sqrt{n_1+1-k} |n_1; a\rangle\langle n_1+1 : a| \right). \tag{3.18}$$

Due to the duality relation between the  $U(2)$  gauge group and  $U(1)$  gauge group on the two-dimensional noncommutative plane [28]-[29], it is possible to establish an isomorphism between Hilbert spaces

$$|n_1 : a\rangle = |2n_1 + a\rangle. \tag{3.19}$$

Therefore  $\tilde{A}_k$ , being a connection of  $U(2)$  gauge group, can be mapped to a  $U(1)$  connection, due to (3.19):

$$A_k^{dual} = \sum_{a=0}^1 \left( \sum_{n_1=0}^{k-1} c_{2n_1+a} |2n_1 + a\rangle\langle 2n_1 + a| + \sum_{n_1=k}^{\infty} \sqrt{n_1+1-k} |2n_1 + a\rangle\langle 2n_1 + 2 + a| \right)$$

$$\begin{aligned}
&= \sum_{n_1=0}^{k-1} (c_{2n_1} |2n_1\rangle\langle 2n_1| + c_{2n_1+1} |2n_1+1\rangle\langle 2n_1+1|) \\
&+ \sum_{n_1=k}^{\infty} \sqrt{n_1+1-k} (|2n_1\rangle\langle 2n_1+2| + |2n_1+1\rangle\langle 2n_1+3|).
\end{aligned} \tag{3.20}$$

Passing from  $n_1 \rightarrow \frac{n_1}{2}$  we obtain

$$\begin{aligned}
A_k^{dual} &= \sum_{n_1=0}^{2k-1} c_{n_1} |n_1\rangle\langle n_1| + \sum_{\frac{n_1}{2}=k}^{\infty} \sqrt{\frac{n_1}{2} - k + 1} (|n_1\rangle\langle n_1+2| \\
&+ |n_1+1\rangle\langle n_1+3|).
\end{aligned} \tag{3.21}$$

While the second part is a gauge transformation of

$$\sum_{n_1=2k}^{\infty} \sqrt{n_1 - 2k + 1} |n_1\rangle\langle n_1+1| \tag{3.22}$$

and therefore equivalent to it, the first part produces into the action an  $U(1)$  instanton of type  $2k$ .

In general, by the same reasoning, a  $U(n)$   $k$ -instanton should be equivalent to a  $n \cdot k$  instanton on  $U(1)$  on the two-dimensional plane.

## 4 Instantons on a four-dimensional noncommutative plane

The generalization of the two-dimensional hermitian matrix model (3.7) on a four-dimensional noncommutative plane is given by the equation

$$S = -\frac{1}{4g^2} \text{Tr}([X_i, X_j] - i\Theta_{ij}^{-1})^2 \tag{4.1}$$

where it is necessary to require that the coordinate commutator matrix  $\Theta_{ij}$  is invertible, i.e.

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij} \quad \det\Theta_{ij} \neq 0. \tag{4.2}$$

Instead of using the commutation relations (4.2) in full generality, we now show how to diagonalize (4.2) in a form which is basically solved by two types of independent raising and lowering oscillator operators.

Firstly we consider the commutation relations (4.2), rewritten in this form:

$$\begin{aligned} [\hat{x}_1, \hat{x}_2] &= i\tilde{\theta}_1 \cos\phi_1 & [\hat{x}_1, \hat{x}_3] &= i\tilde{\theta}_1 \sin\phi_1 \cos\phi_2 & [\hat{x}_1, \hat{x}_4] &= i\tilde{\theta}_1 \sin\phi_1 \sin\phi_2 \\ [\hat{x}_2, \hat{x}_3] &= i\theta_{23} & [\hat{x}_2, \hat{x}_4] &= i\theta_{24} & [\hat{x}_3, \hat{x}_4] &= i\theta_{34}. \end{aligned} \quad (4.3)$$

Let us define a new set of orthogonal cartesian coordinates :

$$\begin{aligned} x'_1 &= x_1 \\ x'_2 &= x_2 \cos\phi_1 + x_3 \sin\phi_1 \cos\phi_2 + x_4 \sin\phi_1 \sin\phi_2 \\ x'_3 &= x_3 \sin\phi_2 - x_4 \cos\phi_2 \\ x'_4 &= x_4 \sin\phi_2 \cos\phi_1 + x_3 \cos\phi_2 \cos\phi_1 - x_2 \sin\phi_1. \end{aligned} \quad (4.4)$$

Then the first part of the commutation relations (4.2) reduces to:

$$[x'_1, x'_2] = i\tilde{\theta}_1 \quad [x'_1, x'_3] = 0 \quad [x'_1, x'_4] = 0. \quad (4.5)$$

After the orthogonal transformation (4.4), the other commutation relations read

$$[x'_2, x'_3] = i\tilde{\theta}_2 \sin\delta_1 \sin\delta_2 \quad [x'_2, x'_4] = i\tilde{\theta}_2 \sin\delta_1 \cos\delta_2 \quad [x'_3, x'_4] = i\tilde{\theta}_2 \cos\delta_1. \quad (4.6)$$

Now we apply another orthogonal transformation:

$$\begin{aligned} x''_1 &= x'_1 \\ x''_2 &= x'_2 \\ x''_3 &= x'_3 \sin\delta_2 + x'_4 \cos\delta_2 \\ x''_4 &= x'_4 \sin\delta_2 - x'_3 \cos\delta_2 \end{aligned} \quad (4.7)$$

to reduce the commutation relations to the form:

$$\begin{aligned}
[x_2'', x_3''] &= i\tilde{\theta}_2 \sin \delta_1 \\
[x_2'', x_4''] &= 0 \\
[x_3'', x_4''] &= i\tilde{\theta}_2 \cos \delta_1.
\end{aligned} \tag{4.8}$$

Finally, we apply a linear non-orthogonal transformation, which is invertible if  $\Theta_{ij}$  is a non degenerate matrix, given by :

$$\begin{aligned}
\tilde{x}_1 &= x_1'' \\
\tilde{x}_2 &= x_2'' \\
\tilde{x}_3 &= \cos \delta_1 x_3'' - \sin \delta_1 x_4'' \\
\tilde{x}_4 &= x_4''
\end{aligned} \tag{4.9}$$

to find:

$$\begin{aligned}
[\hat{x}_1, \hat{x}_2] &= i\theta_1 & [\hat{x}_1, \hat{x}_3] &= 0 & [\hat{x}_1, \hat{x}_4] &= 0 \\
[\hat{x}_2, \hat{x}_3] &= 0 & [\hat{x}_2, \hat{x}_4] &= 0 & [\hat{x}_3, \hat{x}_4] &= i\tilde{\theta}_2 \cos^2 \delta_1 = i\theta_2.
\end{aligned} \tag{4.10}$$

This diagonal form of commutation relations will be used in the following part of the section to define a general class of four-dimensional instanton solutions.

To make contact with noncommutative Yang-Mills theory we decompose the matrices  $X_i$  as a sum of background plus fluctuations as follows:

$$\begin{aligned}
X_1 &= \frac{1}{\theta_1} \tilde{x}_2 + A_1 & X_2 &= -\frac{1}{\theta_1} \tilde{x}_1 + A_2 \\
X_3 &= \frac{1}{\theta_2} \tilde{x}_4 + A_3 & X_4 &= -\frac{1}{\theta_2} \tilde{x}_3 + A_4
\end{aligned} \tag{4.11}$$

and the action turns out to be

$$S = \frac{1}{4g^2} \text{Tr} F_{ij}^2 \quad F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j]. \tag{4.12}$$

The vacuum is again characterized by the solution

$$\begin{aligned}
X^1 + iX^2 &= -\frac{i}{\theta_1}(x^1 + ix^2) = -i\sqrt{\frac{2}{\theta_1}}a_1 = -i\sqrt{\frac{2}{\theta_1}}\left(\sum_{n_1, n_2=0}^{\infty} \sqrt{n_1+1}|n_1, n_2\rangle\langle n_1+1, n_2|\right) \\
X^3 + iX^4 &= -\frac{i}{\theta_2}(x^3 + ix^4) = -i\sqrt{\frac{2}{\theta_2}}a_2 = -i\sqrt{\frac{2}{\theta_2}}\left(\sum_{n_1, n_2=0}^{\infty} \sqrt{n_2+1}|n_1, n_2\rangle\langle n_1, n_2+1|\right)
\end{aligned} \tag{4.13}$$

where

$$[a_1, \bar{a}_1] = [a_2, \bar{a}_2] = 1 \quad [a_1, a_2] = [a_1, \bar{a}_2] = 0. \tag{4.14}$$

The definition, commonly accepted, of soliton is given by

$$U^\dagger X^i U \tag{4.15}$$

obtained by a quasi-unitary operator  $U$  that satisfies

$$UU^\dagger = 1 \quad U^\dagger U = 1 - P_0 \tag{4.16}$$

where  $P_0$  is a finite projection operator.

A simple consequence of (4.16) is that, for an instanton solution we must study the following system of commutation rules

$$[X_i, X_j] = i\Theta_{ij}^{-1}(1 - P_0) \tag{4.17}$$

and give the general and explicit solution of this system for a generic finite projection operator  $P_0$ .

The instanton solution we look for is therefore given by a deformation of the two raising and lowering operators  $a_i (i = 1, 2)$  defined through

$$\begin{aligned}
X_1 + iX_2 &= -i\sqrt{\frac{2}{\theta_1}}A_1 \\
X_3 + iX_4 &= -i\sqrt{\frac{2}{\theta_2}}A_2
\end{aligned} \tag{4.18}$$

and the postulated commutation relations for  $A_i$  are:

$$[A_1, \overline{A}_1] = [A_2, \overline{A}_2] = 1 - P_0 \quad [A_1, A_2] = [A_1, \overline{A}_2] = 0. \quad (4.19)$$

Notice that, due to the duality relation between the tensorial product of Hilbert spaces and the one-oscillator Hilbert space [28]-[29]

$$\mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \quad (4.20)$$

based on the fact that a couple of numbers can be made isomorphic to a number, for example

$$(n_1, n_2) \rightarrow \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} + n_2 \quad (4.21)$$

it is possible to relate the  $U(1)$  four-dimensional connection  $(A_1, A_2)$  to a couple of two-dimensional connections of the  $U(1)$  theory in two dimensions.

Let us define two new quantum numbers :

$$\begin{aligned} n &= n_1 + n_2 \\ k &= n_2 \end{aligned} \quad (4.22)$$

and introduce a short notation for the state :

$$|n_1, n_2\rangle = \left| \frac{n(n+1)}{2} + k \right\rangle = |n; k\rangle. \quad (4.23)$$

A basis of the two-dimensional Hilbert space is determined by the states

$$|n; k\rangle \quad 0 \leq k \leq n \quad \forall n \in \mathbb{N}. \quad (4.24)$$

In this paper however we will need to continue the notation (4.23) to states with  $k \geq n$ , keeping in mind then the equivalence relation

$$|n; k\rangle = |n+1; k-n-1\rangle. \quad (4.25)$$

In the two-dimensional basis, the generic finite projection operator  $P_0$  can be represented in the following form, up to an isomorphism of the Hilbert space:

$$P_0 = |0\rangle\langle 0| + \dots + |m-1\rangle\langle m-1| \quad (4.26)$$

which represents a configuration with instanton number  $m$ .

In the two-dimensional basis (4.24) it is not difficult to derive the quasi-unitary operator  $U$  that produces the projection operator  $P_0$ :

$$\begin{aligned} UU^\dagger &= 1 & U^\dagger U &= 1 - P_0 \\ U &= \sum_{n=0}^{\infty} \sum_{k=0}^n |n; k\rangle\langle n; k+m| \\ U^\dagger &= \sum_{n=0}^{\infty} \sum_{k=0}^n |n; k+m\rangle\langle n; k|. \end{aligned} \quad (4.27)$$

In order to define the instanton configuration, we need to compute

$$A_1^{(1)} = U^\dagger a_1 U \quad A_2^{(1)} = U^\dagger a_2 U \quad (4.28)$$

and it is necessary to represent the lowering oscillator operator  $a_1, a_2$  in the new basis as follows:

$$\begin{aligned} a_1 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sqrt{n+1-k} |n; k\rangle\langle n+1; k| \\ a_2 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sqrt{1+k} |n; k\rangle\langle n+1; k+1|. \end{aligned} \quad (4.29)$$

Computing the products (4.28) is then an easy task, and the result is :

$$\begin{aligned} A_1^{(1)} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sqrt{n+1-k} |n; k+m\rangle\langle n+1; k+m| \\ A_2^{(1)} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sqrt{1+k} |n; k+m\rangle\langle n+1; k+m+1|. \end{aligned} \quad (4.30)$$

An obvious consequence of their definition is that they satisfy the commutation rules (4.19)

$$[A_1^{(1)}, \overline{A}_1^{(1)}] = [A_2^{(1)}, \overline{A}_2^{(1)}] = 1 - P_0 \quad [A_1^{(1)}, A_2^{(1)}] = [A_1^{(1)}, \overline{A}_2^{(1)}] = 0. \quad (4.31)$$

For completeness we give the following formulas:

$$\begin{aligned} A_1^{(1)} A_2^{(1)} &= A_2^{(1)} A_1^{(1)} = \sum_{n=0}^{\infty} \sum_{k=0}^n \sqrt{(n+1-k)(1+k)} |n; k+m\rangle \langle n+2; k+m+1| \\ A_1^{(1)} \overline{A}_2^{(1)} &= \overline{A}_2^{(1)} A_1^{(1)} = \sum_{n=1}^{\infty} \sum_{k=1}^n \sqrt{k(n+1-k)} |n; k+m\rangle \langle n; k+m-1| \\ A_1^{(1)} \overline{A}_1^{(1)} - \overline{A}_1^{(1)} A_1^{(1)} &= A_2^{(1)} \overline{A}_2^{(1)} - \overline{A}_2^{(1)} A_2^{(1)} = \sum_{n=0}^{\infty} \sum_{k=0}^n |n; k+m\rangle \langle n; k+m| = 1 - P_0. \end{aligned} \quad (4.32)$$

The equations of motion are also satisfied, since

$$[X_j, [X_i, X_j]] = 0 \quad \Rightarrow \quad [A_i^{(1)}, P_0] = 0 \quad i = 1, 2 \quad (4.33)$$

as one can immediately verify.

This is the basic solution. It is possible to add to the solution (4.30) an arbitrary configuration that, commuting with everything, doesn't alter the commutation rules (4.19) and the equations of motion (4.33) :

$$\begin{aligned} A_1 &= A_1^{(0)} + A_1^{(1)} & A_2 &= A_2^{(0)} + A_2^{(1)} \\ A_1^{(0)} &= \sum_{i=0}^{m-1} c_i^1 |i\rangle \langle i| & A_2^{(0)} &= \sum_{i=0}^{m-1} c_i^2 |i\rangle \langle i|. \end{aligned} \quad (4.34)$$

In order to derive the configuration  $A_1, A_2$  in terms of the equivalent basis  $|n_1, n_2\rangle$ , we need to pull back the duality between two-dimensional and four-dimensional plane. The task is cumbersome in the general case, and we will perform it only in the simplest case, i.e. a configuration with instanton number  $m = 1$ .

In that case one decomposes, for example,

$$A_1^{(1)} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \sqrt{n+1-k} |n; k+1\rangle \langle n+1; k+1| +$$



$$\begin{aligned}
& + \sum_{n=0}^{\infty} |n, n+1\rangle \langle n+1, n+1| \\
\mapsto \quad & A_1^{(1)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sqrt{n_1+2} |n_1, n_2+1\rangle \langle n_1+1, n_2+1| + \\
& + \sum_{n_1=n_2=0}^{\infty} |n_1+1, 0\rangle \langle 0, n_2+1|.
\end{aligned} \tag{4.35}$$

Analogously for the other connection

$$\begin{aligned}
A_2^{(1)} & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sqrt{n_2+1} |n_1, n_2+1\rangle \langle n_1, n_2+2| + \\
& + \sum_{n_1=n_2=0}^{\infty} \sqrt{1+n_2} |n_1+1, 0\rangle \langle n_1+2, 0|,
\end{aligned} \tag{4.36}$$

Then the quasi-unitary operator looks like:

$$\begin{aligned}
U & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1+1, n_2\rangle \langle n_1, n_2+1| + \sum_{n_1=n_2=0}^{\infty} |0, n_2\rangle \langle n_1+1, 0| \\
U^\dagger & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2+1\rangle \langle n_1+1, n_2| + \sum_{n_1=n_2=0}^{\infty} |n_1+1, 0\rangle \langle 0, n_2| \\
& .
\end{aligned} \tag{4.37}$$

One can check that the commutation rules (4.19) are again satisfied since

$$\begin{aligned}
A_1^{(1)} \bar{A}_1^{(1)} - \bar{A}_1^{(1)} A_1^{(1)} & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2+1\rangle \langle n_1, n_2+1| + \sum_{n_1=0}^{\infty} |n_1+1, 0\rangle \langle n_1+1, 0| \\
& = 1 - |0, 0\rangle \langle 0, 0| = 1 - P_0 = A_2^{(1)} \bar{A}_2^{(1)} - \bar{A}_2^{(1)} A_2^{(1)} \\
A_1^{(1)} A_2^{(1)} = A_2^{(1)} A_1^{(1)} & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sqrt{(n_1+2)(n_2+1)} |n_1, n_2+1\rangle \langle n_1+1, n_2+2| + \\
& + \sum_{n_1=n_2=0}^{\infty} \sqrt{1+n_2} |n_1+1, 0\rangle \langle 0, n_2+2| \\
A_1^{(1)} \bar{A}_2^{(1)} = \bar{A}_2^{(1)} A_1^{(1)} & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sqrt{(n_1+2)(n_2+1)} |n_1, n_2+2\rangle \langle n_1+1, n_2+1| + \\
& + \sum_{n_1=n_2=0}^{\infty} \sqrt{1+n_2} |n_1+2, 0\rangle \langle 0, n_2+1|.
\end{aligned} \tag{4.38}$$

Another possible generalization is extending our solution to  $d = 2n$  noncommutative planes. By diagonalizing the  $\Theta_{ij}$  matrix as a sum of independent  $n$ -oscillators, it is not difficult to generalize our trick. For example let us consider the six-dimensional case. The basic coordinates commutation relations are:

$$\begin{aligned} [\hat{x}_1, \hat{x}_2] &= i\theta_1 \\ [\hat{x}_3, \hat{x}_4] &= i\theta_2 \\ [\hat{x}_5, \hat{x}_6] &= i\theta_3. \end{aligned} \tag{4.39}$$

Introducing three independent raising and lowering oscillator operators  $A_i, i = 1, 2, 3$  through the relations :

$$\begin{aligned} X_1 + iX_2 &= -i\sqrt{\frac{2}{\theta_1}}A_1 \\ X_3 + iX_4 &= -i\sqrt{\frac{2}{\theta_2}}A_2 \\ X_5 + iX_6 &= -i\sqrt{\frac{2}{\theta_3}}A_3 \end{aligned} \tag{4.40}$$

the six-dimensional instanton is defined through the commutation relations :

$$\begin{aligned} [A_1, \bar{A}_1] &= [A_2, \bar{A}_2] = [A_3, \bar{A}_3] = 1 - P_0 \\ [A_1, A_2] &= [A_1, \bar{A}_2] = [A_1, A_3] = [A_1, \bar{A}_3] = [A_2, A_3] = [A_2, \bar{A}_3] = 0. \end{aligned} \tag{4.41}$$

By using again the duality between

$$\mathcal{H} \times \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \tag{4.42}$$

based on the fact that three numbers can be made isomorphic to a number

$$(n_1, n_2, n_3) \rightarrow n = f(n_1, n_2, n_3) \tag{4.43}$$

one generalizes the solution (4.30) to six dimensions.

## 5 Conclusion

In this paper we have found a general class of soliton solutions starting from the Matrix model approach instead of the twistor approach of Nekrasov and Schwarz. We believe that this intrinsic characterization of solitons within the matrix model is necessary to simplify the classification of instantons and to study quantum mechanics around this nonperturbative solutions. For example the contribution to the partition function of gauge theory from an instanton can be easily derived as an ordinary matrix integral of the fluctuations around the configuration.

We have started our study from the unitary matrix model, where the soliton solutions have been characterized as the only solutions of the matrix model which survive in the  $N \rightarrow \infty$  limit and give a finite contribution to the action.

As a byproduct of this investigation we have reduced a  $U(n)$  soliton in terms of a  $U(1)$  soliton in two dimensions, generalizing the well known Morita equivalence of the noncommutative torus to the noncommutative plane, but we believe that this can be done in every even dimensions.

Then the solitons within the unitary matrix model have been compared with the solitons defined in the hermitian matrix model, finding as characterization the following equation

$$[X_i, X_j] = i\Theta_{ij}^{-1}(1 - P_0). \quad (5.1)$$

This equation has been analyzed in detail in  $d = 2n$  dimensions, and given the general solution, consistent with the equation of motion

$$[X_j, [X_i, X_j]] = 0. \quad (5.2)$$

The construction of the soliton solutions in  $d = 2n$  planes is based on representing the gauge connection as a sum of two parts, whose role is to build a projection operator in the commutation rules (5.1).

We believe that this systematic point of view will help in classifying all the possible instanton solutions, giving an accurate and simple scheme of the nonperturbative properties of the noncommutative gauge theory in  $d = 2n$ .

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